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OPTIMAL LOT-SIZE DETERMINATION FOR A TWO WARE-HOUSE PROBLEM WITH DETERIORATION AND SHORTAGES USING NPV

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Abstract—In this paper, we develop a inventory model for deteriorating items with two warehouses (assuming deterioration rates in the two-warehouses to differ) by minimizing the net present value (NPV) of the total cost. We allow for shortages and complete backlogging and prove here that the optimal replenishment policy not only exists but also is conditionally unique. Further, the result reveals that the reorder interval based on the average total cost, if it exists, must be longer than that derived using NPV. Finally, using a numerical example we illustrate the model and conclude the article with suggestions for possible future research.

Index Terms— Deterioration; Discounted cash flow; NPV; Shortages; Two-warehouse;

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1 INTRODUCTION

In classical EOQ model, a single owned warehouse (OW) with unlimited capacity is usually assumed. But situations like temporary price discounts or bulk purchases or seasonal product's availability etc. make retailers buy warehouses or rebuild a new warehouse. The manager always finds it economically beneficial to maintain a rented warehouse (RW) in addition to the owned one. This will also make the system replenish more goods than can be stored in own warehouse. Hence inventory models should be extended to the situation with multiple warehouses.

Again, the effect of deterioration is very important in many inventory systems. Deterioration is defined as decay or damage such that the item cannot be used for its original purpose. Most of the physical goods undergo decay or deterioration over time. Commodities such as fruits, vege tables, food stuffs, etc. suffer from depletion by direct spoilage while kept in store.

Highly volatile liquids such as gasoline, alcohol, turpentine etc. undergo physical depletion with time through the process of evaporation. Blood, electronic goods, radioactive substances, photographic films, food grains etc. deteriorate through a gradual loss of potential or utility with time. Thus, decay or deterioration of physical goods in stock is a very realistic feature.

In recent years, various researchers have discussed a two-warehouse inventory system. In formulation of the basic economic order quantity (EOQ) model, the demand rate of the item was assumed to be constant. Ghare et al. (1963) were the first researchers to consider the effect of deterioration on inventory items. They derived on economic order quantity (EOQ) model where inventory items decay exponentially with time. A two-warehouse inventory system was first proposed by Hartely(1976). There it was assumed that the holding cost in RW is greater than that in OW. Hence items in RW were first transferred to OW to meet the demand until the stock level in RW drops to zero and then items in OW are released. Goswami et al. (1992) further developed the model with or without shortages by assuming that the demand varies over time with linearly increasing trend and that the transportation

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cost from RW to OW depends on the quantity being transported. Pakkala et al. (1992) extended the two-warehouse inventory model for deteriorating items with finite replenishment rate and shortages.

The ideas of time-varying demand for deteriorating items with two storage facilities were also considered by Benkherouf (1997) and Bhunia et al. (1998). Goyal et al. (2001) presented a review of deteriorating inventory literature of the early 1990s. Murdeshwar et al. (1985) extended this model to the case of finite replenishment rate. Dave (1988) further discussed the cases of bulk release pattern for both finite and infinite replenishment rates. He rectified the errors in Murdeshwar et al. [1985] and gave a complete solution for the model given by Sharma (1983). Pakkala et al. (1992) extended the two-warehouse inventory model for deteriorating items with finite replenishment rate and shortages, taking time as discrete and continuous variable, respectively. In these models mentioned above, the demand rate was assumed to be constant. Subsequently, the ideas of time-varying demand and stock-dependent demand were considered by some authors, such as Benkherouf (1997), Goswami et al. (1998), Bhunia et al. (1998), Kar et al. (2001) and others. Yang (2004) proposed an alternative model for determining the optimal replenishment cycle for the two-warehouse inventory problem under inflation, in which the inventory deteriorates at a constant rate over times and shortages were allowed . By assuming that the inventory system will operate for a long time, he determined the optimal values of the decision variables by minimizing the average total cost.

Zhou (2003) presented a multi-warehouse inventory model for non-perishable items with time-varying demand and partial backlogging. Abad (1996, 2001) discussed a pricing and lot-sizing problem for a product with a variable rate of deterioration, allowing shortages and partial backlogging. Since the deterioration depends on preserving facilities and environmental conditions available in a warehouse, different warehouses may have different deterioration rates. As deterioration phenomenon is taken into account, a unit of inventory stored incurs holding cost and deterioration cost. However, we can determine the decision variables by minimizing the discounted value of all future costs (i.e. NPV of total cost). Hadley (1964) compared the optimal order quantities determined by minimizing these two different objective functions. When the discount rate is excessive, he obtained the optimal reorder intervals with significant differences for these two models. Rachamadugu (1988) developed error bounds for EOQ model by minimizing net present value approximately. Sun et al. (2002) investigated the general multiproduct, production and inventory model using the NPV of the total cost as the objective function stating that the reorder interval based on the average total cost could be much longer than that derived using NPV.

In this paper, we develop a deterministic inventory model for deteriorating items with two-warehouses. We allow for shortages and complete backlogging, and assume that the inventory costs including holding cost and deterioration cost in RW is higher than that in OW. The firm stores goods in OW before RW, but clears the stocks in RW before OW. However, we minimize the NPV of the total cost. For generality, the deterioration rate in RW is different from one in OW. Due to consideration towards the effect of the discount rate, which relates to the purchasing power of money, purchasing cost must be included. We compute the purchase cost instead of the deterioration cost and obtain the condition which gives the unique solution and develop the criterion to find the optimal replenishment policy. We then compare the decision using the NPV with one using the average total cost. The result reveals that the reorder interval based on the average total cost, if it exists, must be longer than that derived using NPV.

2 NOTATION AND ASSUMPTIONS:

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2.1 Assumptions

In order to develop the mathematical model of the two warehouse inventory replenishment policy, the assumptions adopted in this paper are as below:

1. Lead time is zero.
2. Replenishment rate is infinite.
3. Shortages are allowed and completely backlogged
4. The time horizon of the inventory system is infinite.
5. The own warehouse (OW) has a fixed capacity of W units and the rented warehouse (RW) has unlimited capacity.
6. The goods of OW are consumed only after consuming the goods kept in RW, i.e. $T = t_r + t_s$.
7. The unit inventory costs (including holding cost and deterioration cost) per unit time in RW are higher than those in OW; i.e. $C_{hr} + \beta C > C_{ho} + \alpha C$

2.2 Notation

In addition, the following assumptions are imposed:

D demand rate per unit time

A replenishment cost per order

C purchasing cost per unit

r discount rate

C_{ho} holding cost per unit per unit time in OW

C_{hr} holding cost per unit per unit time in RW

C_s backorder cost per unit per unit time

a deterioration rate of the stored commodity

α deterioration rate in OW, where $0 \le \alpha < 1$

β deterioration rate in RW, where $0 \le \beta < 1$

W capacity of the own warehouse

Q ordering quantity per cycle

B maximum inventory level per cycle

t_r length of period during which the inventory level reaches zero in RW

t_o length of period during which the inventory level reaches zero in OW

t_s length of period during which shortages are allowed

T length of the inventory cycle

$I_r(t)$ level of positive inventory in RW at time t

$I_o(t)$ level of positive inventory in OW at time t

$I_s(t)$ level of negative inventory at time t

$TC(t_r, t_s)$ net present value of cash flows for the first cycle

$NPV(t_r, t_s)$ net present value of total cost

$ACT(t_r, t_s)$ average total cost

3 Mathematical Formulation

We have considered here the traditional shortage model associated with a two-warehouse inventory problem. It starts with an instant replenishment and ends with shortages as in the figure. The ordering quantity over the replenishment cycle can be determined as follows:

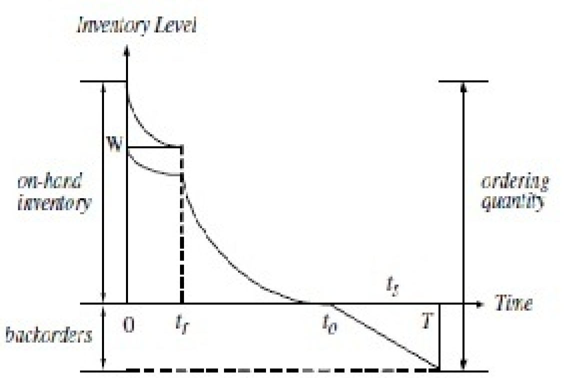

Fig. 1. Graphical representation of a two-warehouse inventory system.

In a rented ware house

$$dI_r(t)/dt + \beta I_r(t) = -(D + aI_r(t)),$$
$$0 < t < t_r,\ I_r(t_r) = 0 \qquad \text{.... (1)}$$

$$I_r(t) = (D/(\alpha+\beta))(e^{(\alpha+\beta)(t_r - t)} - 1) \qquad \text{.... (2)}$$

In the own warehouse

$$dI_o(t)/dt + \alpha I_o(t) = 0$$

$$0 < t < t_r,\ I_o(0) = W \qquad .... (3)$$

$$I_o(t) = We^{-\alpha t} \qquad .... (4)$$

Again

$$dI_o(t)/dt + \alpha I_o(t) = -(D + aI_o(t)),$$

$$t_r < t < t_o,\ I_o(t_o) = 0 \qquad .... (5)$$

$$I_o(t) = (D/(a+\beta))(e^{(\alpha+\beta)(t_r - t)} - 1) \qquad .... (6)$$

Both in the rented warehouse and in own warehouse

$$dI_s(t)/dt = -D,\ t_o < t < t_o + t_s = T,$$

$$I_s(t_o) = 0 \quad .... (7)$$

$$I_s(t) = -D(t_o - t) \qquad .... (8)$$

The ordering quantity over the replenishment cycle is

$$Q = I_r(0) + I_o(0) - I_s(T)$$
$$= I_r(t)(D/(a+\beta))(e^{(a+\beta)t_r} - 1) + W + Dt_s \qquad .... (9)$$

The maximum inventory level per cycle is

$$B = I_r(0) + I_o(0)$$
$$= (D/(a+\beta))(e^{(a+\beta)t_r} - 1) + W \qquad .... (10)$$

With an instantaneous cash transaction during sales, the present value of purchase cost for the first cycle can be obtained as

$$C[I_r(0) + I_o(0) + e^{-r(t_o+t_s)} I_s(t_o + t_s)]$$

$$= (D/(a+\beta))(e^{(a+\beta)t_r} - 1) + W + e^{-r(t_o+t_s)} Dt_s$$

Holding cost in the rented warehouse

$$C_{hr}\int_0^{t_r} I_r(t)dt$$

$$= (C_{hr}D/(r.(a+\alpha).(r+a+\beta))[re^{(a+\beta).t_r}$$
$$+ (a+\beta).e^{-rt_r} - (r+a+\beta)]$$

Holding cost in own warehouse

$$C_{ho}\left[\int_0^{t_r} I_o(t)dt + \int_{t_r}^{t_o} I_o(t)dt\right]$$

$$= (C_{ho}.W/(a+\alpha)+r) + (C_{ho}.D/r(r+a+\alpha)).(e^{-rt_o} - e^{-rt_s})$$

Backorder cost (shortage cost)

$$C_s\int_{t_o}^{T} -I_s(t)dt$$

$$= (C_s D/r^2).e^{-r(t_o+t_s)}.(e^{rt_s} - rt_s - 1)$$

Replenishment cost $= A_o$

Total cash flow TC (t_r, t_s)

$$= A_o + C\left[(D/(a+\beta)).(e^{(a+\beta)t_r} - 1) + W + e^{-r(t_o+t_s)}Dt_s\right] +$$

$$(C_{hr}.D/r.(a+\beta).(r+a+\beta)).\left[re^{(a+\beta)t_r} + (a+\beta)e^{-rt_r} - (r+a+\beta)\right]$$

$$+ (C_{ho}.W/(a+\alpha)+r)) + (C_{ho}.D/r.(a+\alpha+r))(e^{-rt_o} - e^{-rt_s})$$

$$+ (C_s.D/r^2)e^{-r(t_o+t_s)}(e^{rt_s} - rt_s - 1) \qquad .... (11)$$

Due to the continuity of $I_o(t)$ at $t = t_r$ from (4) and (6) we get

$$We^{-t(a+\alpha)} = (D/(a+\alpha))(e^{(a+\alpha).(t_o - t_r)} - 1)$$

This implies

$$t_o = t_r + (1/(a+\alpha))\ln\{((a+\alpha)W/D).(e^{-(a+\alpha).t_r} + 1)\}$$

.... (12)

where t_o is a function of t_r and

$$dt_o/dt_r = \left(1/(1 + ((a+\alpha)We^{-(a+\alpha)t_r}/D))\right) < 1$$

i.e. $dt_o/dt_r - 1 < 0$

Let $NPV(t_r, t_s)$ net present value of total cost over the horizon [0,∞)

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Then $NPV(t_r,t_s)=\sum_{n=0} TC(t_r,t_s)e^{-nr(t_o+t_s)}$

$$=TC(t_r,t_s)\sum_{n=0} e^{-nr(t_o+t_s)}$$

$$=TC(t_r,t_s)\left(1/\left(1-e^{-r(t_o+t_s)}\right)\right) \quad \text{.... (13)}$$

We determine t_r and t_s such that $NPV(t_r,t_s)$ is minimized

$$\partial NPV(t_r,t_s)/\partial t_r = dt_o \Big/ dt_r \begin{bmatrix} \left(-re^{-r(t_o+t_s)}\Big/\left(1-e^{-r(t_o+t_s)}\right)^2\right)TC(t_r,t_s) \\ +\left(1/\left(1-e^{-r(t_o+t_s)}\right)\right).\left(1/(dt_o/dt_r)\right).\left(\partial TC(t_r,t_s)/\partial t_r\right) \end{bmatrix}$$

..... (14)

$$\partial NPV(t_r,t_s)/\partial t_s = \begin{bmatrix} \left(-re^{-r(t_o+t_s)}\Big/\left(1-e^{-r(t_o+t_s)}\right)^2\right)TC(t_r,t_s) \\ +\left(1/\left(1-e^{-r(t_o+t_s)}\right)\right).\left(1/\left(1-e^{-r(t_o+t_s)}\right)\right).\left(\partial TC(t_r,t_s)/\partial t_s\right) \end{bmatrix}$$

..... (15)

Where, $\dfrac{\partial TC(t_r,t_s)}{\partial t_r}=\left\{\dfrac{D}{1+\left((a+\alpha)We^{-(a+\alpha)t_r}\Big/D\right)}\right\}$ x

$$\left\{C\left[e^{rt_o}-rt_s e^{-r(t_o+t_s)}\right]+e^{rt_o}K(t_r)-\frac{C_s}{r}e^{-r(t_o+t_s)}(e^{rt_s}-rt_s-1)\right\}$$

..... (16)

$$\frac{\partial TC(t_r,t_s)}{\partial t_s}=De^{-r(t_o+t_s)}\left[C+(C_s-rc)t_s\right] \quad \text{..... (17)}$$

$$K(t_r)=\frac{C_{hr}(r+a+\beta)C}{(r+a+\beta)}\left[e^{(a+\beta)t_r+rt_o}-e^{-r(t_o-t_r)}\right]$$

$$\left(1+\frac{(a+\alpha)We^{-(a+\alpha)t_r}}{D}\right)+$$

$$\frac{C_{ho}+(r+a+\alpha)C}{(r+a+\alpha)}\left[e^{r(t_o-t_r)}-1\right]+$$

$$\frac{C_{ho}+(r+a+\alpha)C}{(r+a+\alpha)}\times e^{r(t_o-t_r)}.\frac{(a+\alpha)We^{-(a+\alpha)t_r}}{D} \quad \text{..... (18)}$$

and $\dfrac{dt_o}{dt_r}$ is defined as

$$\frac{dt_o}{dt_r}=\frac{1}{1+\left\{\dfrac{(a+\alpha)We^{-(a+\alpha)t_r}}{D}\right\}}<1 \quad \text{(from 12)}$$

The optimal solution of (t_r,t_s) must satisfy

$\dfrac{\partial NPV(t_r,t_s)}{\partial t_r}=0$ and $\dfrac{\partial NPV(t_r,t_s)}{\partial t_s}=0$ simultaneously.

Then from (14) and (15)

$$-re^{-r(t_o+t_s)}TC(t_r,t_s)=\frac{1-e^{-r(t_o+t_s)}}{\dfrac{dt_o}{dt_r}}\frac{\partial TC(t_r,t_s)}{\partial t_r} \quad \text{..... (19)}$$

$$re^{-r(t_o+t_s)}TC(t_r,t_s)=\left[1-e^{-r(t_o+t_s)}\right]\frac{\partial TC(t_r,t_s)}{\partial t_s} \quad \text{...... (20)}$$

Since both left hand sides in (19) and (20) are same, their right hand sides are equal. Comparing them

$$\frac{C_s-rC}{r}(1-e^{-rt_s})=K(t_r) \quad \text{...... (21)}$$

We substitute $TC(t_r,t_s)$ in (11) and $\dfrac{\partial TC(t_r,t_s)}{\partial t_s}=0$ in

equations (17) and (20), and obtain

$$D[1-e^{-r(t_o+t_s)}][C+(C_s-rC)t_s]$$

$$=r\left\{A+C\left[\frac{D}{a+\beta}(e^{(a+\beta)t_r}-1)+W\right]\right.$$

$$+\frac{C_{hr}.D}{r(a+\beta)(r+a+\beta)}\begin{bmatrix} re^{(a+\beta)t_r}+(\alpha+\beta)e^{-rt_r} \\ -(r+a+\beta) \end{bmatrix}$$

$$+\frac{C_{h0}.W}{(a+\alpha)+r}+\frac{C_{h0}.D}{r(r+a+\alpha)}\left(e^{-rt_0}-e^{-rt_s}\right)$$

$$\left.+\frac{C_S.D}{r^2}e^{-r(t_0+t_s)}\left(e^{rt_s}-rt_s-1\right)\right\}$$

..... (22)

LEMMA 1

If $D > (a+\alpha)W$, then *K(t_r)* is continuous and is strictly increasing function of $t_r \in [0,\infty)$ and its range is

$$\left[\left(\frac{C_{ho}+(r+a+\alpha)C}{(r+a+\alpha)}\right)\left\{(1+\frac{(a+\alpha)W}{D})^{\frac{r}{a+\alpha}+1}-1\right\},\infty\right]$$

Proof:

It is obvious that $K(t_r)$ is a continuous function of $t_r \in [0,\infty)$. Next taking the derivative of $K(t_r)$ with respect to t_r

$$\frac{dK(t_r)}{dt_r}=\left[C_{hr}+(r+a+\beta)C\,\frac{(a+\alpha)W\left(e^{-(a+\alpha)t_r}\right)}{D}e^{[(a+\alpha)t_r+rt_o]}\right]$$
$$\times\left\{\frac{De^{(a+\alpha)t_r}}{(a+\alpha)W}-\frac{\alpha-\beta}{r+a+\beta}(1-e^{-(r+a+\beta)t_r})\right\}$$
$$+[(C_{hr}+(\alpha+\beta)C)-(C_{ho}+(a+\alpha))C]\times\frac{(a+\alpha)We^{-(a+\alpha)t_r}}{D}e^{r(t_o-t_r)}$$

Let,

$$H(t_r)=\frac{De^{(a+\alpha)t_r}}{(a+\alpha)W}-\frac{\alpha-\beta}{r+a+\beta}(1-e^{-(r+a+\beta)t_r}),t_r\ge 0$$

Thus,

$$\frac{dH(t_r)}{dt_r}=\frac{De^{(a+\alpha)t_r}}{(a+\alpha)W}-(\alpha-\beta)e^{-(r+a+\beta)t_r}$$
$$>\frac{D}{W}-(a+\alpha)e^{-(r+a+\beta)t_r}$$
$$=(a+\alpha)[\frac{D}{(a+\alpha)W}-e^{-(r+a+\beta)t_r}]$$
$$>(a+\alpha)[\frac{D}{(a+\alpha)W}-1]$$

If $D>(a+\alpha)W$ then we know that $\frac{dH(t_r)}{dt_r}>0$.

Therefore, $H(t_r)$ is a strictly increasing function in the interval $[0,\infty)$ which implies

$$H(t_r)>H(0)=\frac{D}{(a+\alpha)W}>0 \text{ for } t_r>0$$

From the above result and assumption (13),

$\frac{dK(t_r)}{dt_r}>0$ for $t_r>0$, Therefore, $K(t_r)$ is a strictly increasing function in the interval $[0,\infty)$.

$$K(0)=\left(\frac{C_{ho}+(r+a+\alpha)C}{(r+a+\alpha)}\right)\left\{(1+\frac{(a+\alpha)W}{D})^{\frac{r}{a+\alpha}+1}-1\right\}$$

and $\lim_{t_r\to\infty}K(t_r)=\infty$ are trivial.

From lemma 1, to guarantee that the optimal solution exists, we assume that D is larger than the maximum deteriorating quantity for the items in OW, $(a+\alpha)W, i.e. D>(a+\alpha)W$. This result is obvious. Thus, from now onwards we assume that $D>(a+\alpha)W$ in this article.

LEMMA 2

If $K(0)\ge\frac{C_s-rC}{r}$ then the nonnegative solution of (t_r,t_s) which satisfies equation (21) does not exist.

Proof:

If $K(0)\ge\frac{C_s-rC}{r}$, then

$K(t_r)>(1-e^{-rt_s})\left(\frac{C_s-rC}{r}\right)$ for $t_s\in[0,\infty)$. On the other hand, from Lemma 1, we have, $K(t_r)$ is strictly increasing function of $t_r\in[0,\infty)$. Thus, a value of t_r cannot be found in the interval $[0,\infty)$ such that $K(t_r)>(1-e^{-rt_s})\left(\frac{C_s-rC}{r}\right)$. This completes the proof.

From lemma 2, we see that the optimal solution exists only if $\frac{C_s-rC}{r}>K(0)$. When the inequality $\frac{C_s-rC}{r}>K(0)$

holds, equation implies that t_s is a function of $t_r \in [0,\infty)$. Taking the partial derivative of both sides in equation (21) with respect to t_r, it gives

$$(C_s - rC)e^{-rt_s}\frac{dt_s}{dt_r} = \frac{dK(t_r)}{dt_r} > 0 \qquad \text{..... (23)}$$

Thus, we obtain $\frac{dt_s}{dt_r} > 0$. From lemma 1, $K(t_r)$ is a continuous and strictly increasing function of $t_r \in [0,\infty)$, thus we can find a unique value $t^{\wedge}{}_r \in [0,\infty)$ such that $K(t_r^{\wedge}) = \frac{C_s - rC}{r}$.

Moreover, since both t_r and t_s must be nonnegative, the feasible solution for t_r which satisfies in equation (21) should be chosen in the interval $[0,t_r^{\wedge})$. Therefore, we can obtain the following result: once we get the optimal value $t_r^* \in [0,t_r^{\wedge})$, the optimal solutions of t_o and t_s (denoted by t_o* and t_s*, respectively) can be uniquely determined by equations (12) and (21) respectively, and given as follows:

$$t_o^* = t_r^* + \frac{1}{a+\alpha}\ln\left\{1 + \frac{(a+\alpha)W}{D}\left(e^{-(a+\alpha)t^*{}_r}\right)\right\} \text{ and}$$

$$t_s^* = \frac{1}{r}\ln\frac{C_s - rC}{C_s - rC - rK(t_r^*)} \qquad \text{..... (24)}$$

Now we can derive the optimal value t_r*. Using equation (22), we let

$$\begin{aligned}G(t_r) &= D[1 - e^{-r(t_o+t_s)}][C + (C_s - rC)t_s] \\ &- r\left\{A + C\left[\frac{D}{a+\beta}(e^{(a+\beta)t_r} - 1) + W\right]\right. \\ &+ \frac{C_{hr}.D}{r(a+\beta)(r+a+\beta)}\begin{bmatrix} re^{(a+\beta)t_r} + (\alpha+\beta)e^{-rt_r} \\ -(r+a+\beta)\end{bmatrix} \\ &+ \frac{C_{h0}.W}{(a+\alpha)+r} + \frac{C_{h0}.D}{r(r+a+\alpha)}\left(e^{-rt_0} - e^{-rt_s}\right) \\ &+ \frac{C_S.D}{r^2}e^{-r(t_0+t_s)}\left(e^{rt_s} - rt_s - 1\right)\end{aligned}$$

$$\text{when } t_r \in [0,t_r^{\wedge}) \qquad \text{..... (25)}$$

After assembling equation (21) and (23), the first derivative of $G(t_r)$ with respect to $t_r \in \left[0,t_r{}^{\wedge}\right)$ $t_r \in [0,t_r^{\wedge})$, becomes

$$\frac{dG(t_r)}{dt_r} = D(C_S - rC)\left[1 - e^{-r(t_0+t_s)}\right]\frac{dt_s}{dt_r} > 0$$

Therefore, $G(t_r)$ is a strictly increasing function in the interval $[0,t_r^{\wedge})$ because $\lim_{t_r \to t_r^{\wedge} - t_s} G(t_r) = \infty$, it results

$$\begin{aligned}&\lim_{t_r \to t_r^{\wedge} - t_s} G(t_r) = \lim_{t_s \to \infty}\{D[C + (C_s - rC)t_s] \\ &- r\{A + C\left(\frac{D}{a+\beta}(e^{(a+\beta)t_r} - 1) + W\right) \\ &+ \frac{C_{hr}.D}{r(a+\beta)(r+a+\beta)}\begin{bmatrix} re^{(a+\beta)t_r} + (\alpha+\beta)e^{-rt_r} \\ -(r+a+\beta)\end{bmatrix} \\ &+ \frac{C_{h0}.W}{(a+\alpha)+r} + \frac{C_{h0}.D}{r(r+a+\alpha)}\left(e^{-rt_0} - e^{-rt_s}\right) \\ &+ \frac{C_S.D}{r^2}e^{-r(t_0)}\}\end{aligned}$$

Then we have the following result.

LEMMA 3

For any given $\frac{C_s - rC}{r} > K(0)$, we have

(a) If $G(0) \le 0$, then the solution $t_r^* \in [0,t_r^{\wedge})$, which satisfies equation (22) not only exists but is also unique.

(b) If G(0)>0, then the solution $t_r^* \in [0,t_r^{\wedge})$, which satisfies equation (22) does not exist.

Proof:

(a) First we consider G(0)<0. Since $G(t_r)$ is a strictly increasing function in $[0,t_r^{\wedge})$ and $\lim_{t_r \to t_r^{\wedge}} G(t_r) = \infty$,by using initial value theorem , there exists a unique solution $t_r^* \in [0,t_r^{\wedge})$ such that *$G(t_r^*)=0$* i.e. t_r* is the unique solution that satisfies equation (22).

Next if *G(0)=0*, then from the property that $G(t_r)$ is strictly increasing in *$[0,t_r^{\wedge})$*, then *$t_r^*=0$* is the unique value which satisfies *$G(t_r^*)=0$*. In this case, the inventory system reduces to the own warehouse problem.

(b) From the property that *$G(t_r)$* is strictly increasing in

[0,tr^), if *G(0)>0*, then *G(tr)>0* $G(t_r) > 0, \forall t_r \in [0, t_r^{\wedge})$. Thus we can find a value $t_r^* \in [0, t_r^{\wedge})$ such that *G(tr*)=0.*

THEOREM 1

For any given $\dfrac{C_s - rC}{r} > K(0)$, we have

(a) If $G(0) < 0$, then the point (t_r^*, t_s^*) which satisfies equation (21) and (22) simultaneously, and $t_r^* \in [0, t_r^{\wedge})$ is the global minimum point of the net present value of total cost .

(b) If $G(0) \geq 0$, then the optimal $t_r^* = 0$ in this case, the inventory system reduces to one- warehouse problem.

Proof:

(a) $G(0) < 0$

Since $NPV(t_r, t_s) = TC(t_r, t_s) / (1 - e^{-r(t_o + t)_s})$

The necessary conditions for minimum are

$$\partial NPV(t_r, t_s)/\partial t_r = (dt_o/dt_r)\left[\begin{array}{l} \dfrac{-re^{-r(t_0+t_s)}}{[1-e^{-r(t_0+t_s)}]^2} TC(t_r, t_s) \\ + \dfrac{1}{[1-e^{-r(t_0+t_s)}]} \times \dfrac{1}{(dt_o/dt_r)} \times \dfrac{\partial TC(t_r, t_s)}{\partial t_s} \end{array}\right]$$

$$\frac{\partial NPV(t_r, t_s)}{\partial t_s} = \frac{-re^{-r(t_0+t_s)}}{[1-e^{-r(t_0+t_s)}]^2} TC(t_r, t_s) + \frac{1}{[1-e^{-r(t_0+t_s)}]} \times \frac{\partial TC(t_r, t_s)}{\partial t_s} = 0$$

This implies

$$\left[\frac{1}{\frac{dt_0}{dt_r}} \frac{\partial TC(t_r, t_s)}{\partial t_r}\right]_{[(t_r,t_s)=(t_r^*,t_s^*)]} = \left[\frac{\partial TC(t_r, t_s)}{\partial t_s}\right]_{(t_r,t_s)=(t_r^*,t_s^*)}$$

$$= \left[\frac{re^{-r(t^*_0 + t^*_s)}}{[1-e^{-r(t^*_0+t^*_s)}]^2} TC(t^*_r, t^*_s)\right]$$

where t_o^* is defined as in equation (24).

From Lemma 3(a), the solution $t_r^* \in [0, t_r^{\wedge})$ which satisfies (22) not only exists but is also unique. Hence, the value of t_s^* can be uniquely determined by equation (24).

Again,

$$\left[\frac{\partial^2 NPV(t_r, t_s)}{\partial t_r^2}\right]_{(t_r,t_s)=(t_r^*,t_s^*)} = [D(C_s - rC)\frac{e^{-r(t_0+t_s)}}{1-e^{-r(t_0+t_s)}} \frac{dt_0}{dt_r}\frac{dt_s}{dt_r}]_{(t_r,t_s)=(t_r^*,t_s^*)} > 0$$

$$\left[\frac{\partial^2 NPV(t_r, t_s)}{\partial t_s^2}\right]_{(t_r,t_s)=(t_r^*,t_s^*)}$$

$$= \left[[D(C_s - rC)\frac{e^{-r(t_0+t_s)}}{1-e^{-r(t_0+t_s)}}]_{(t_r,t_s)=(t_r^*,t_s^*)}\right] > 0$$

And

$$\left[\frac{\partial^2 NPV(t_r, t_s)}{\partial t_r \partial t_s}\right]_{(t_r,t_s)=(t_r^*,t_s^*)}$$

$$= [\frac{1}{1-e^{-r(t_0+t_s)}}\{rDe^{-r(t_0+t_s)}[C + (C_s - rC)t_s]\frac{dt_0}{dt_r}$$

$$- rDe^{-r(t_0+t_s)}[C + (C_s - rC)t_s]\frac{dt_0}{dt_r}\}]_{(t_r,t_s)=(t_r^*,t_s^*)} = 0$$

stationary point (t_r^*, t_s^*) is

$$\det(H) = \left[\frac{\partial^2 NPV(t_r, t_s)}{\partial t_r^2}\right]_{(t_r,t_s)=(t_r^*,t_s^*)} \times \left[\frac{\partial^2 NPV(t_r, t_s)}{\partial t_s^2}\right]_{(t_r,t_s)=(t_r^*,t_s^*)} - \left[\frac{\partial^2 NPV(t_r, t_s)}{\partial t_r t_s}\right]_{(t_r,t_s)=(t_r^*,t_s^*)}$$

Hence, the hessian matrix H at point (t_r^*, t_s^*) is positive definite. We conclude that the stationary point for our optimization problem is a global minimum point.

(b) For $G(0) = 0$, from the proof of lemma 3(a), $t_r^* = 0$ is the unique solution which satisfies $G(t_r^*) = 0$

For $G(0) > 0$, by equation (18), (21) and (25), equation (14) becomes

$$\frac{\partial NPV(t_r,t_s)}{\partial t_r}=\frac{dt_0}{dt_r}\{\frac{-re^{-r(t_0+t_s)}}{[1-e^{-r(t_0+t_s)}]^2}TC(t_r,t_s)$$
$$+\frac{1}{1-e^{-r(t_0+t_s)}}\frac{1}{\frac{dt_0}{dt_r}}\frac{\partial TC(t_r,t_s)}{\partial t_r}\}$$
$$=\frac{e^{-r(t_0+t_s)}}{[1-e^{-r(t_0+t_s)}]^2}G(t_r)\frac{dt_0}{dt_r}$$

Because $dt_o/dt_r>0$ and $G(t_r)$ is a strictly increasing function, we have $\partial NPV(t_r,t_s)/\partial t_r>0$ for any $t_r\in[0,t_r^{\wedge})$ which implies that for any fixed $t_s\in[0,\infty)$, a smaller value of t_r causes a lower value of $NPV(t_r,t_s)$. As a result, the minimum value of $NPV(t_r,t_s)$ occurs at the boundary point $t_r^*=0$.

For the special circumstance where $t_r^*=0$, since the RW is not used, the model reduces to the one-warehouse inventory problem. This completes the proof.

From theorem 1(a), once the optimal solution (t_r^*,t_s^*) is obtained, we substitute (t_r^*,t_s^*) into equations (9) and (13), the optimal ordering quantity per cycle, Q^*, and the minimum net present value of total cost $NPV(t_r^*,t_s^*)$ are as follows:

$$Q^*=\frac{D}{a+\beta}(e^{(a+\beta)t_r^*}-1)+W+Dt_se^{-r(t_o^*+t_s^*)}$$

and

$$NPV(t_r^*,t_s^*)=\frac{D}{r}[C+(C_s-rC)t_s^*] \quad ..... (26)$$

From the special case that $t_r^*=0$ in theorem 1(b), the model reduces to one-warehouse inventory problem. Let $C_{hr}=C_{ho}$, $\beta=\alpha$ and $W=0$. We can obtain the objective function from equation (11). Then the optimal solution of the one- ware house inventory problem can be solved by using similar arguments. Next, we want to compare the decision using the net present value with one using the total average cost, let $ATC(t_r,t_s)$ be the average total cost, then we have

$$ATC(t_r,t_s)=\frac{TC(t_r,t_s)}{t_o+t_s} \quad ..... (27)$$

Solving the necessary conditions $\frac{\partial ATC(t_r,t_s)}{\partial t_r}=0$ and $\frac{\partial ATC(t_r,t_s)}{\partial t_s}=0$

For the minimum value of $ATC(t_r,t_s)$ we get

$$((C_s-rC)/r)(1-e^{-rt_s})=K(t_r) \quad ..... (28)$$

and

$$\begin{aligned}&D(t_o+t_s)e^{-r(t_o+t_s)}[C+(C_s+rC)t_s]\\&=A+C[(D/(\alpha+\beta))(e^{(\alpha+\beta)t_r}-1)+W]\\&+(C_{hr}D/(r(a+\beta)(r+a+\beta)))[re^{(a+\beta)t_r}\\&+(a+\beta)e^{-rt_r}-(r+a+\beta)]\\&+C_{ho}W/((a+\alpha)+r)+C_{ho}D/(r(r+a+\alpha))(e^{-rt_0}-e^{-rt_s})\\&+(C_SD/r^2)e^{-r(t_0+t_s)}(e^{rt_s}-rt_s-1)\end{aligned} \quad ...... (29)$$

It is obvious that equation (28) is the same as equation (21). Arguing similarly as the previous section, if equation (28) holds, then we have $(C_s-rC)/r>K(0)$ and t_s is a function of t_r, where $t_r^{\wedge}\in[0,\infty)$ and satisfies $K(t_r^{\wedge})=(C_s-rC)/r$ once we get the optimal value $t_r^{**}\in[0,t_r^{\wedge})$, the optimal solutions of t_o and t_s denoted by t_o^{**} and t_s^{**} respectively can be uniquely determined. Next motivated by equation (21) we let

$$Z(t_r)=D(t_o+t_s)e^{-r(t_o+t_s)}[C+(C_s+rC)t_s]-A$$
$$+C\left[\frac{D}{a+\beta}\left(e^{(a+\beta)t_r}-1\right)+W\right]$$

$$+\frac{C_{hr}.D}{r(a+\beta)(a+r+\beta)}\begin{bmatrix} re^{(a+\beta)t_r}+(a+\beta)e^{-rt_r} \\ -(a+r+\beta) \end{bmatrix}$$
$$+\frac{C_{ho}.W}{(a+\alpha)+r}+\frac{C_{ho}.D}{r(r+a+\alpha)}\left(e^{-rt_o}-e^{-rt_s}\right)$$
$$+\frac{C_sD}{r^2}e^{-r(t_o+t_s)}.\left(e^{rt_s}-rt_s-1\right) \quad ..... (30)$$

Since, $1-e^{-r(t_o+t_s)}=[e^{r(t_o+t_s)}-1]e^{-r(t_o+t_s)}>r(t_o+t_s)e^{-r(t_o+t_s)}$ we obtain that $G(t_r)>rZ(t_r)$ for all $t_r>0$ Therefore if there exists a value t_r^{**} such that $Z(t_r^{**})=0$ must be larger than the value t_r^* where $G(t_r^*)=0$. Summarizing the above arguments we have the following result.

Proposition 1

If the solution of $t_r^{**}\in[0,t_r^{\Lambda})$, which satisfies $Z(t_r)=0$, exists, then $t_r^{**}>t_r^*$.

From Proposition 1, if t_r^{**} exists then it is easily seen that $t_o^{**}>t_o^*$ and $t_s^{**}>t_s^*$ i.e. the length of the inventory cycle based on the average cost is longer than one based on NPV.

4 Numerical Example

To illustrate the above results, we consider this example: D = 400, W = 100, C= 10, $C_{ho}=0.2$, $C_{hr}=0.5$, $C_s=2$, a = 0.01, $\alpha=0.01$, $\beta=0.04$, r = 0.06 in appropriate units. Then, the numerical results for $NPV(t_r,t_s)$ and $ATC(t_r,t_s)$ are shown in the table below.

Table 1.Numerical results for $NPV(t_r,t_s)$ and $ATC(t_r,t_s)$

A	t_r^*	t_o^*	t_s^*	Q*	$NPV(t_r^*,t_s^*)$
100	0.1875	0.4359	0.4052	337.4	70447.6
250	0.1875	0.8063	0.0348	339.2	66991.5
	r^{**}	t_o^{**}	t_s^{**}	Q**	$ATC(t_r^{**},t_s^{**})$
100	0.7619	1.0075	1.1417	867.3	3433.4
	3.3637	3.5969	5.8132	3890.6	5332.5
250	0.7619	1.3746	0.7756	870.8	11463.1
	3.3637	3.9446	5.5555	3937.4	5136.6

Table 2.Sensitivity analysis of $NPV(t_r^*,t_s^*)$

r	Δ	$NPV(t_r^*,t_s^*)$	c	Δ	$NPV(t_r^*,t_s^*)$
.02	89.8496	210632.0	5	28.1822	36817.6
.04	39.799	105551.0	10	23.0815	70448.5
.08	14.697	52838.2	15	17.9807	103789.2
.10	9.6456	42224.6	20	12.8799	135494.3
β	Δ	$NPV(t_r^*,t_s^*)$	C_{hr}	Δ	$NPV(t_r^*,t_s^*)$
0.04	23.081	70447.6	1.0	23.0563	70504.8
0.09	23.081	70505.3	2.5	23.0059	70590.7
0.24	23.081	70591.2	5.0	22.9555	70646.2
α	Δ	$NPV(t_r^*,t_s^*)$	C_s	Δ	$NPV(t_r^*,t_s^*)$
0.00	23.117	70382.4	0.5	-1.92	--
0.01	23.081	70447.6	1.0	6.41	69244.9
0.04	23.011	70631.8	2.0	23.08	70447.6

5 Concluding Remarks

In this paper, an inventory model is developed for deteriorating items with two levels of storage, permitting shortage and complete backlogging. In particular, we use the NPV of total cost as the objective function for the generalized inventory system. The general pattern analytical formulations of the problem has been given. The condition which guarantees that the unique solution exists, is obtained and the complete proof of corresponding second-order sufficient conditions for optimum. It will be economical to consume the goods of RW at the earliest. However, this does not mean that the firm always takes time to search a preserving facility with a lower deterioration rate than that in OW. Hence, the condition,

$D > (a+\alpha)W$ is more suitable than the assumption of deterioration rates in RW and OW. In addition, when the discount rate r is small, we have,

$$1-e^{-r(t_o+t_s)} = e^{-r(t_o+t_s)}[e^{r(t_o+t_s)}-1] \approx r(t_o+t_s)e^{-r(t_o+t_s)}$$

This implies $G(t_r) \approx rZ(t_r)$. Hence, the optimal solution based on average total cost will be a good approximation to the one based on NPV.

The proposed model can be extended in several ways. Firstly, we can easily extend the backlogging rate of unsatisfied demand to any decreasing function $\beta(x)$ where x is the waiting time up to the next replenishment, and $0 \le \beta(x) \le 1$ with $\beta(0)=1$ Secondly, we can also incorporate the quantity discount and the leaning-curve phenomenon into the model.

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